

The compatibility of thin-shell wormholes with quantum field theory

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Abstract

It is shown in this paper that thin-shell wormholes, mathematically constructed by the standard cut-and-paste technique, can, under fairly general conditions, be compatible with quantum field theory.

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1 Introduction

Wormholes are handles or tunnels in the geometry of spacetime connecting different regions of our Universe or of different universes. The pioneer work of Morris and Thorne [1] has shown that macroscopic wormholes may not only exist but may actually be traversable by humanoid travelers. Wormholes can only be held open by the use of “exotic” matter. Such matter violates the weak energy condition.

While wormholes are predictions of Einstein’s theory, quantum field theory places severe restrictions on the existence of traversable wormholes [2, 3, 4, 5]. In fact, Ford and Roman [4, 5] have shown that the wormholes discussed in Ref. [1] could not exist on a macroscopic scale. The wormholes in Refs. [6] and [7] could in principle exist, but they are subject to extreme fine-tuning. Fine-tuning is also required in Refs. [8] and [9], where a suitable extension of the quantum inequalities of Ford and Roman is used to show that Morris-Thorne wormholes can be compatible with quantum field theory by striking a balance between the size of the exotic region and the concomitant fine-tuning of the metric coefficients. Ref. [10] continues the theme by showing that a relatively small modification of the charged wormhole model of Kim and Lee [11, 12] can make such a wormhole compatible with quantum field theory.

A powerful method for describing or mathematically constructing a type of spherically symmetric wormholes from black-hole spacetimes was proposed by Visser [13]. Known as *thin-shell wormholes*, they are constructed by the so-called cut-and-paste technique since

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their construction requires grafting two black-hole spacetimes together. The thin shell is actually the junction surface.

It is proposed in this paper that under fairly general conditions a thin-shell wormhole can be compatible with quantum field theory.

2 Traversable wormholes

The spacetime geometry of a static, spherically symmetric Lorentzian wormhole can be described by the metric (using units in which $c = G = 1$)

$$ds^2 = -e^{2\Phi(r)} dt^2 + \frac{dr^2}{1 - b(r)/r} + r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (1)$$

where $\Phi(r)$ and $b(r)$ have continuous derivatives. It is usually assumed that $\Phi(r) \rightarrow 0$ and $b(r)/r \rightarrow 0$ as $r \rightarrow \infty$, i.e., the spacetime is asymptotically flat. (This condition will be automatically satisfied due to some later conditions.) The function $\Phi(r)$ is called the *redshift function*, which must be everywhere finite to prevent an event horizon. We are also going to require that $\Phi'(r) > 0$. The function $b(r)$ is called the *shape function* since it determines the spatial shape of the wormhole when viewed, for example, in an embedding diagram. The minimum radius $r = r_0$ is called the *throat* of the wormhole, where $b(r_0) = r_0$. Finally, $b'(r_0) \leq 1$, referred to as the *flare-out condition* in Ref. [1].

Next, let us recall that the Einstein field equations $G_{\hat{\alpha}\hat{\beta}} = 8\pi T_{\hat{\alpha}\hat{\beta}}$ imply that in the orthonormal frame, the components of the stress-energy tensor are proportional to the components of the Einstein tensor. The only nonzero components are $T_{\hat{t}\hat{t}} = \rho$, $T_{\hat{r}\hat{r}} = p$, and $T_{\hat{\theta}\hat{\theta}} = T_{\hat{\phi}\hat{\phi}} = p_r$, where ρ is the energy density, p the radial pressure, and p_r the lateral pressure.

To hold a wormhole open, the weak energy condition (WEC) must be violated. The WEC states that the stress-energy tensor $T_{\hat{\alpha}\hat{\beta}}\mu^{\hat{\alpha}}\mu^{\hat{\beta}}$ must obey

$$T_{\hat{\alpha}\hat{\beta}}\mu^{\hat{\alpha}}\mu^{\hat{\beta}} \geq 0 \quad (2)$$

for all time-like vectors and, by continuity, all null vectors. Using the radial outgoing null vector $\mu^{\hat{\alpha}} = (1, 1, 0, 0)$, the condition becomes $T_{\hat{t}\hat{t}} + T_{\hat{r}\hat{r}} = \rho + p \geq 0$. So if the WEC is violated, then $\rho + p < 0$. Matter that violates the WEC is referred to as “exotic.”

From the Einstein field equations, the components of the stress-energy tensor turn out to be [1, 13]

$$\rho = \frac{1}{8\pi} \frac{b'}{r^2}, \quad (3)$$

$$p = \frac{1}{8\pi} \left[-\frac{b}{r^3} + 2 \left(1 - \frac{b}{r} \right) \frac{\Phi'}{r} \right], \quad (4)$$

$$p_t = \frac{1}{8\pi} \left(1 - \frac{b}{r} \right) \left[\phi'' + \Phi' \left(\Phi' + \frac{1}{r} \right) \right] - \frac{1}{2} \left(\frac{b}{r} \right)' \left(\Phi' + \frac{1}{r} \right). \quad (5)$$

Of particular interest in this paper is the *thin-shell wormhole* first proposed by Visser [13]. The theoretical construction is also referred to as the cut-and-paste technique and

is now considered standard. While it is possible to start with two copies of a generic geometry, Visser initially confined the construction to Schwarzschild spacetimes. In similar manner, we will take two copies of an asymptotically flat black-hole spacetime with (outer) event horizon $r = r_h$ and remove from each the four-dimensional region

$$\Omega^\pm = \{r \leq a | a > r_h\}.$$

The construction proceeds by identifying (in the sense of topology) the time-like hypersurfaces

$$\partial\Omega^\pm = \{r = a | a > r_h\}.$$

The resulting manifold is geodesically complete and consists of two asymptotically flat regions connected by a throat, thereby forming a traversable Lorentzian wormhole.

The reason for our interest in thin-shell wormholes is the following key property: all the exotic matter is confined to an infinitely thin shell. The property is, of course, an idealization: we assume that the shell is extremely thin compared to the radius of the throat.

3 The quantum inequalities

In this section we briefly discuss the quantum inequalities due to Ford and Roman [5], slightly extended in [8, 9]. These inequalities constrain the magnitude and time duration of negative energy and, as a consequence, place severe restrictions on Lorentzian wormholes. The starting point is an inertial Minkowski spacetime without boundaries. So if u^ν is the tangent vector to a timelike geodesic, then $\langle T_{\mu\nu} u^\mu u^\nu \rangle$ is the expectation value of the local energy density in the observer's frame of reference. So if τ is the observer's proper time and τ_0 the duration of the sampling time, then

$$\frac{\tau_0}{\pi} \int_{-\infty}^{\infty} \frac{\langle T_{\mu\nu} u^\mu u^\nu \rangle d\tau}{\tau^2 + \tau_0^2} \geq -\frac{3}{32\pi^2 \tau_0^4}. \quad (6)$$

According to Ref. [5], the energy density is sampled in a time interval of duration τ_0 which is centered around an arbitrary point on the observer's worldline so chosen that $\tau = 0$ at this point. The inequality is valid in curved spacetime as long as τ_0 is small compared to the local proper radii of curvature. Applied to spherically symmetric traversable wormholes in Ref. [1], it was found that none were able to meet this condition, i.e., the throat sizes could only be slightly larger than Planck length.

For the purpose of discussing wormholes, more convenient forms of the above quantum inequality can be obtained. To do so, we need the following length scales modeled after the length scales in Ref. [5], introduced in Ref. [8]:

$$r_m \equiv \min \left[r, \left| \frac{b(r)}{b'(r)} \right|, \frac{1}{|\Phi'(r)|}, \left| \frac{\Phi'(r)}{\Phi''(r)} \right| \right]. \quad (7)$$

The reason is that the components of the Riemann curvature tensor can be expressed in the following form:

$$R_{\hat{r}\hat{t}\hat{r}\hat{t}} = \left(1 - \frac{b(r)}{r}\right) \frac{1}{\frac{\Phi'(r)}{\Phi''(r)} \frac{1}{\Phi'(r)}} - \frac{b(r)}{2r} \left(\frac{1}{\frac{1}{\Phi'(r)} \frac{b(r)}{b'(r)}} - \frac{1}{r \frac{1}{\Phi'(r)}} \right) + \left(1 - \frac{b(r)}{r}\right) \frac{1}{\left(\frac{1}{\Phi'(r)}\right)^2}, \quad (8)$$

$$R_{\hat{\theta}\hat{t}\hat{\theta}\hat{t}} = R_{\hat{\phi}\hat{t}\hat{\phi}\hat{t}} = \left(1 - \frac{b(r)}{r}\right) \frac{1}{r \frac{1}{\Phi'(r)}}, \quad (9)$$

$$R_{\hat{\theta}\hat{r}\hat{\theta}\hat{r}} = R_{\hat{\phi}\hat{r}\hat{\phi}\hat{r}} = \frac{b(r)}{2r} \left(\frac{1}{r \frac{b(r)}{b'(r)}} - \frac{1}{r^2} \right), \quad (10)$$

and

$$R_{\hat{\theta}\hat{\phi}\hat{\theta}\hat{\phi}} = \frac{1}{r^2} \frac{b(r)}{r}. \quad (11)$$

The purpose of these length scales is to obtain an upper bound for R_{\max} , the maximum curvature: observe that the largest value of $(1 - b(r))/r$ and of $b(r)/r$ is unity. So disregarding the coefficient $1/2$, we conclude that $R_{\max} \leq 1/r_m^2$. Moreover, the smallest radius of curvature r_c is

$$r_c \approx \frac{1}{\sqrt{R_{\max}}} \geq r_m.$$

On this scale the spacetime is approximately Minkowskian, so that inequality (6) can be applied with an appropriate τ_0 .

Returning to wormholes, it is proposed in Ref. [5] that the static frame be replaced by a “boosted frame,” i.e., by Lorentz transforming to a frame of a radially moving geodesic observer moving with velocity v relative to the static frame. In this boosted frame the energy density ρ' may be negative, so that inequality (6) can be applied. In this boosted frame (the “primed system”)

$$r'_c \approx \frac{1}{\sqrt{R'_{\max}}} \geq \frac{r_m}{\gamma}, \quad (12)$$

where $\gamma = (1 - v^2)^{-1/2}$. The suggested sampling time is

$$\tau_0 = \frac{f r_m}{\gamma} \ll r'_c, \quad (13)$$

where f is a scale factor such that $f \ll 1$. The energy density in the boosted frame is

$$\rho' = T_{\hat{0}'\hat{0}'} = \gamma^2 T_{\hat{t}\hat{t}} + \gamma^2 v^2 T_{\hat{r}\hat{r}} = \gamma^2 (\rho + v^2 p), \quad (14)$$

where v is the velocity of the boosted observer. It is stated in Ref. [5] that the energy density does not change very much over the short sampling time considered here, so that $\rho' \geq -3/(32\pi^2\tau_0^4)$ is approximately constant. From Eqs. (3) and (4),

$$\rho' = \frac{\gamma^2}{8\pi r^2} \left[b'(r) - v^2 \frac{b(r)}{r} + v^2 r (2\Phi'(r)) \left(1 - \frac{b(r)}{r} \right) \right].$$

For ρ' to be negative, v has to be sufficiently large:

$$v^2 > \frac{b'(r)}{\frac{b(r)}{r} - 2r\Phi'(r) \left(1 - \frac{b(r)}{r}\right)}. \quad (15)$$

In particular, at the throat, $v^2 > b'(r)$. Given $b(r)$, inequality (15) places a restriction on $\Phi'(r)$.

Next, from

$$\frac{3}{32\pi^2\tau_0^4} \geq -\rho'$$

we have

$$\frac{32\pi^2\tau_0^4}{3} \leq \frac{8\pi r^2}{\gamma^2} \left[v^2 \frac{b(r)}{r} - b'(r) - v^2 r (2\Phi'(r)) \left(1 - \frac{b(r)}{r}\right) \right]^{-1}. \quad (16)$$

Using $\tau_0 = fr_m/\gamma$ and dividing both sides by r^4 , we have (disregarding a small coefficient)

$$\frac{f^4 r_m^4}{r^4 \gamma^4} \leq \frac{1}{r^2 \gamma^2} \left[v^2 \frac{b(r)}{r} - b'(r) - 2v^2 r \Phi'(r) \left(1 - \frac{b(r)}{r}\right) \right]^{-1}. \quad (17)$$

Our final step is to insert l_p to obtain a dimensionless quantity:

$$\frac{r_m}{r} \leq \left(\frac{1}{v^2 b(r)/r - b'(r) - 2v^2 r \Phi'(r) (1 - b(r)/r)} \right)^{1/4} \frac{\sqrt{\gamma}}{f} \left(\frac{l_p}{r} \right)^{1/2}. \quad (18)$$

This is the first version of the extended quantum inequality expressed in terms of the boosted frame. At the throat, where $b(r_0) = r_0$, inequality (18) reduces to Eq. (95) in Ref. [5]:

$$\frac{r_m}{r_0} \leq \left(\frac{1}{v^2 - b'(r_0)} \right)^{1/4} \frac{\sqrt{\gamma}}{f} \left(\frac{l_p}{r_0} \right)^{1/2}. \quad (19)$$

At the throat, this inequality is trivially satisfied whenever $b'(r_0) = 1$. It was extended in Ref. [8]. The purpose of the extended inequality (18) is to include the region around the throat, instead of just the throat itself. [See Ref. [8] for details.] Also discussed in [8] is a second inequality that omits both v^2 and γ . The reason is that, according to Ref. [2], the boosted frame can be replaced by a static observer. In this frame, r_m is then replaced by ℓ_m , the proper distance. The extended quantum inequality now has the slightly more convenient form

$$\frac{\ell_m}{r} \leq \left(\frac{1}{b(r)/r - b'(r) - 2r\Phi'(r) (1 - b(r)/r)} \right)^{1/4} \frac{1}{f} \left(\frac{l_p}{r} \right)^{1/2}. \quad (20)$$

4 How much exotic matter?

According to Ford and Roman [5], the exotic matter must be confined to an extremely thin band around the throat. This constraint is amply satisfied for a thin-shell wormhole where the throat surface is theoretically infinitely thin. So we are dealing with an extremely small interval, as a result of which $\rho + p$ is approximately constant, i.e., $\rho + p = -\eta$

($\eta > 0$). To calculate the volume, we need an appropriate volume measure: $dV = 4\pi r^2 dr$ or $\sqrt{g} dr d\theta d\phi$. So if the shell extends from $r = r_0$ to $r = r_1$ (on both sides of the throat), we get

$$2 \int_{r_0}^{r_1} (-\eta) dV = 2 \int_{r_0}^{r_1} (-\eta) (4\pi r^2 dr) = -\frac{8\pi\eta}{3} (r_1^3 - r_0^3). \quad (21)$$

This inequality can be made arbitrarily small by choosing r_1 sufficiently close to r_0 . So if the shell is infinitely thin, then the amount of exotic matter becomes infinitely small. It is noted in Refs. [2, 3, 8], however, that the amount of exotic matter cannot be arbitrarily small, but as already discussed in Sec. 2, an infinitely thin shell is an idealization: the amount of exotic matter could be relatively small but not arbitrarily small.

As noted earlier, having the exotic matter confined to a very thin band is a necessary condition for the existence of a traversable wormhole, but the condition is not sufficient for the sought-after compatibility with quantum field theory. That is the topic of the next section, which deals with wormholes in general, not just the thin-shell type.

5 A volume integral

Measuring the amount of exotic matter by means of a volume integral involving b and Φ was first proposed by Visser *et al.* [14] and continued by Nandi *et al.* [15]. Using Eqs. (3) and (4), it is readily checked that

$$\rho + p = \frac{1}{8\pi r} \left(1 - \frac{b}{r}\right) \left[\ln \left(\frac{e^{2\Phi}}{1 - b/r} \right) \right]'. \quad (22)$$

Using the volume measure $dV = 8\pi r^2 dr$ from the previous section, Eq. (22) becomes

$$\rho + p = (r - b) \left[\ln \left(\frac{e^{2\Phi}}{1 - b/r} \right) \right]'. \quad (23)$$

Integrating by parts, we then obtain

$$\oint (\rho + p) dV = (r - b) \ln \left(\frac{e^{2\Phi}}{1 - b/r} \right) \Big|_{r_0}^{\infty} - \int_{r_0}^{\infty} (1 - b') \left[\ln \left(\frac{e^{2\Phi}}{1 - b/r} \right) \right] dr. \quad (24)$$

Regarding the boundary term, observe that since $b(r_0) = r_0$,

$$\lim_{r \rightarrow r_0} (r - b) \ln(e^{2\Phi}) - (r - b) \ln \left(1 - \frac{b}{r} \right) = 0 - \lim_{r \rightarrow r_0} r \ln \left(1 - \frac{b}{r} \right)^{1-b/r} = -r_0 \ln 1 = 0.$$

It is noted in Ref. [15] that the boundary term also vanishes as $r \rightarrow \infty$ provided that $\Phi(r) \sim \mathcal{O}(r^{-2})$ and $b(r) \sim \mathcal{O}(r^{-1})$, which can likewise be easily confirmed.

The vanishing boundary term leaves

$$\oint (\rho + p) dV = - \int_{r_0}^{\infty} (1 - b') \left[\ln \left(\frac{e^{2\Phi}}{1 - b/r} \right) \right] dr. \quad (25)$$

It is easy to show that this integral is well behaved near $r = r_0$. For the upper limit, the conditions $\Phi(r) \sim \mathcal{O}(r^{-2})$ and $b(r) \sim \mathcal{O}(r^{-1})$ carry the day.

Now recall from the previous section that on the interval $[r_0, r_1]$ the amount of exotic matter can be made as small as desired by shrinking the interval. The only way to obtain a small value for the integral in Eq. (25) is by letting b' be close to unity. For convenience, let us now restate inequality (20):

$$\frac{\ell_m}{r} \leq \left(\frac{1}{b(r)/r - b'(r) - 2r\Phi'(r)(1 - b(r)/r)} \right)^{1/4} \frac{1}{f} \left(\frac{l_p}{r} \right)^{1/2}. \quad (26)$$

Observe that at the throat, where $b(r_0) = r_0$, the inequality is trivially satisfied whenever $b'(r_0) = 1$. Moving away from the throat, the denominator must still be 0 or close to, calling for an appropriate adjustment of the redshift function $\Phi(r)$. Such an adjustment is in principle possible for the following reason: it is shown in Ref. [8] that, for any of the typical shape functions, $b(r)/r - b'(r) > 0$; at the same time, $\Phi'(r) > 0$, an assumption that was brought out earlier. (The only other assumptions made are that $\Phi(r)$ remains finite and that $\Phi(r) \sim \mathcal{O}(r^{-2})$, which have no direct bearing on the problem.) So inequality (26) can be satisfied in the neighborhood of the throat, provided that $b'(r)$ remains close to 1 (while gradually declining). But the small amount of exotic matter due to the thin shell automatically forces b' to be close to 1 near the throat by Eq. (25). The thin-shell wormhole is therefore compatible with quantum field theory under fairly general conditions.

6 The volume integral theorem

The volume integral for determining the amount of exotic matter has been extended to [15]

$$\int_0^{2\pi} \int_0^\pi \int_{r_0}^\infty (\rho + p) \sqrt{-g_4} dR d\theta d\phi, \quad (27)$$

where $R = r - a$. Applied to an infinitely thin shell of radius a , it is sometimes assumed that $\rho = \delta(R)\sigma(a)$, where $\sigma(a)$ is the energy density of the thin shell and $\delta(R)$ the Dirac delta function [16]. Since an infinitely thin shell does not exert any radial pressure, we now have

$$\int_0^{2\pi} \int_0^\pi \int_{-\infty}^\infty \delta(R)\sigma(a)\sqrt{-g_4} dR d\theta d\phi = 4\pi\sigma(a)a^2. \quad (28)$$

For a Schwarzschild spacetime, where

$$\sigma(a) = -\frac{1}{2\pi a} \sqrt{1 - \frac{2M}{a}},$$

the result is

$$\Omega = -2a \sqrt{1 - \frac{2M}{a}};$$

moreover, for large a , $\Omega \approx -2a$. If a is measured in meters, this can lead to a huge amount of exotic matter since we are using geometrized units. So while the use of the

delta function is mathematically correct, when applied to an infinitely thin surface, this use does not appear to be appropriate for expressing the density ρ of the shell.

7 Summary

Before summarizing the results, let us recall our basic assumptions: besides the standard assumptions and definitions for any wormhole, we assumed that $\Phi'(r) > 0$ and that $\Phi(r) \sim \mathcal{O}(r^{-2})$. We also assumed that $b = b(r)$ is a typical shape function in the sense of Ref. [8], implying that $b(r)/r - b'(r) > 0$. It was found that, broadly speaking, thin-shell wormholes can in principle be made to meet the generalized quantum inequalities and therefore become compatible with quantum field theory.

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